Chapter 3 : Construction of general measure:

In Khoshnevisan Ch 3, he does the following:

1) Measure spaces, σ-algebras, algebras.

Algebras and σ-algebras.

- $X \in \mathcal{F}$
- Closed under (countable) unions
- Closed under complements.

2) Definition of measure: $\mu : \mathcal{F} \longrightarrow [0, \infty]$

(i) $\mu(\phi) = 0$

(ii) Countable additivity.

Countable subadditivity : Let $\mu : \mathcal{G} \longrightarrow [0,\infty]$ be a SET-FUNCTION. It is countably subadditive if $\forall \{A_i\}_{i=1}^{\infty} \in \mathcal{G}$, st $\bigcup_{j=1}^{\infty} A_j \in \mathcal{G}$, we have

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \sum_{j=1}^{\infty} \mu(A_j)$$

★Exer: Countably additive fns are countably subadditive.

Pf: NOT true. Suppose $\{A_i\}$ st $\bigcup_{i=1}^{\infty} A_i \in \mathcal{G}$

μ is a set fn on Ω. $\mathcal{G} = \{ A_1, A_2, \cdots, \bigcup_{i=1}^{\infty} A_i \}$

$\mu(\bigcup_{i=1}^{\infty} A_i) = 5$ $\mu(A_i) = \frac{1}{2^i}$. A_i are not disjoint, so nothing to check.

But $\mu(\bigcup_{i=1}^{\infty} A_i) > \sum_{i=1}^{\infty} \mu(A_i)$

If $\mathcal{G}$ is a σ-algebra then its true.

Ok, so lemma 3.9 probably means, let $\mathcal{G}$ be a σ-algebra.

Lemma: 3.11 Let $(\Omega, \mathcal{F}, \mu)$ be a measure space

1) If $A_n \uparrow A$ show $\lim_{n\to\infty} \mu(A_n) = \mu(A)$

2) Same holds if $A_n \downarrow A$ and $\mu(A_n) < \infty$ for some n.

↳ Ex: Construct an example where this doesn't hold if $\mu(A_n)$ not finite for any n.

Examples:

- (Discrete space) $\Omega = \{\omega_i\}_{i=1}^{\infty}$, $\mathcal{F} = 2^{\Omega}$ Suppose

$P(\{\omega_i\}) = p_i$ $\sum_{i=1}^{\infty} p_i < \infty$. Define $P(A) = \sum_{i \in A} p_i$

- (Delta mass): $\delta_x(A) = 1_A(x)$

Construction of General Measures and Lebesgue measure

I will do this a bit differently from Davar to link it up with real analysis.

Review:

Outer measure induced by a set function

$\mu : S \longrightarrow [0,\infty]$ set function on a space X.

$\mu^*(\phi) = 0 \qquad \mu^*(E) = \inf \sum_{k=1}^{\infty} \mu(E_k)$

Then $\mu^* : 2^X \longrightarrow [0,\infty]$ is an outer measure.

<u>countable subadditivity</u>: let $E \subseteq \bigcup_{k=1}^{\infty} E_k$ To show that $\mu^*(E) \leq \sum_{k=1}^{\infty} \mu^*(E_k)$. By definition of outer measure, for each E_k choose $\bigcup_{i=1}^{\infty} E_{ki}$ such that

$\sum_{i=1}^{\infty} \mu(E_{ki}) < \mu^*(E_k) + \frac{\epsilon}{2^k}$.

Then E is covered by $\{E_{ki}\}_{k,i=1}^{\infty}$ and thus

$\mu^*(E) \leq \sum_{k=1}^{\infty} \mu^*(E_k) + \epsilon$

__Definition:__ (Caratheodory Extension)

$\mu : \Sigma \to [0,\infty]$ set function.

μ^* induced outer measure

$\mathcal{F}$ = collection of measurable sets.
$= \{ E \in 2^X : \forall A \in 2^X,\ \mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c) \}$

$\bar{\mu} : \mathcal{F} \to [0,\infty]$ induced Caratheodory meas.

Previously had $\mu : S \to [0,\infty]$ and

$$\begin{tikzcd} & \mu : S \to [0,\infty] \arrow[dl] & \\ \mu^* : 2^X \to [0,\infty] & & \bar{\mu} : \mathcal{F} \to [0,\infty] \arrow[ll, dashed, "\text{restriction}"'] \arrow[ul] \end{tikzcd}$$

where $\mathcal{F}$ is the σ-algebra of measurable sets.

__Question:__ When does $\mathcal{F}$ contain S and $\bar{\mu}|_S = \mu$?

That is, when is $\bar{\mu}$ an EXTENSION of μ?

__Prop__ (Necessary condition): If $\bar{\mu}$ is an extension of μ, then μ must be

1) finitely additive
2) countably subadditivity if $E \subseteq \bigcup_{n=1}^{\infty} E_n \Rightarrow \mu(E) \le \sum_{n=1}^{\infty} \mu(E_n)$
3) $\mu(\phi) = 0$

If this changed to $E = \bigcup_{n=1}^{\infty} E_n$ then there are counterexamples.

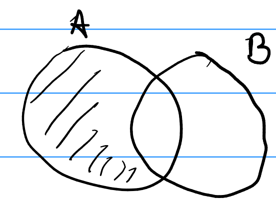
Def: (Premeasure). $\mu: S \to [0, \infty]$ is a premeasure if

1) finitely additive
2) countabley monotone
3) $\mu(\phi) = 0$

Def (Relative complement closure). S is closed wrt relative complements if $A, B \in S \Rightarrow A \setminus B \in S$

Remark: Relative complement closure
$\Rightarrow$ Finite intersection closure.

$A \cap B = A \setminus (A \setminus B)$



Theorem: Suppose S is relative complement closed and μ is a premeasure. Then $\bar{\mu}$ is an extension of μ called Caratheodory.

Remark :

1) finite additivity of μ + Relative complement closure of S
$\Longrightarrow$ every $E \in \mathcal{F}$ was measurable

2) Countable monotonicity of μ
$\iff$ $\mu^*(E) = \mu(E) \quad \forall E \in S$

Let $S = \{$ collection of all intervals on $\mathbb{R}\}$

Is S closed under relative complements ?

(Closed under intersections and relative complements can be written as finite disjoint unions)

Semi Rings : S is a semiring if $\forall A, B \in S$
$A \cap B \in S$ and $\exists \{C_i\}_{i \geq 1}^{k}$ disjoint such that
$A \setminus B = \bigcup_{i \geq 1}^{k} C_i$ (think intervals or boxes)

(Ring of sets) Closed under finite union and relative complement $\Rightarrow$ closed under intersection.
So in fact $\phi \in$ Ring.

Ring of sets $\subseteq$ Semiring of sets

since $A \setminus B$ is a union of 1 set that's already in the ring.

Prop: Let S be a semiring of sets. Let S' = finite disjoint unions of sets in S'. Then, S' is relative complement closed, and if μ is a premeasure on S, it has a unique extension to S' as a premeasure.

σ-finite set functions Space X, collection S. We have μ is σ-finite on X if

$$X = \bigcup_{n=1}^{\infty} S_n \qquad \mu(S_n) < \infty \quad \text{and} \quad S_n \in S$$

Theorem (Caratheodory-Hahn) Let μ be a premeasure on S, where S is a semi-ring. Then $\bar{\mu}$ the induced measure is an extension of μ.

If μ is σ-finite, so is $\bar{\mu}$ and $\bar{\mu}$ is unique.

$$\begin{array}{ccccc}
\bar{\mu} : \mathcal{F} \to [0,\infty] & & \xleftarrow{\text{restriction}} & & \\
\Big\downarrow \text{extension} & & & & \\
\mu : S \to [0,\infty] & \longrightarrow & \mu' : S' \to [0,\infty] & \longrightarrow & \mu^* : 2^X \to [0,\infty] \\
\text{semiring} & & \text{closed under rel complements} & &
\end{array}$$

Corollary Let S be a semiring on X, $\mathcal{B}$ smallest σ-algebra containing S. Then two σ-finite meas agree on $\mathcal{B}$ iff they agree on S!

Remark: Ring of sets that contains X is called an algebra. Semiring of sets " " is called a semialgebra.

Lebesgue measure: Let $\Omega = [0,1]$

Let $S = \{ (a,b], a \leq b \}$, define:

$\mu:(a,b] = b-a$

Prop: μ is a pre-measure.

Pf: 1) $\mu(\phi) = 0$

2) Finite additivity. Let $\{A_i\}_{i=1}^{\infty} \in S$ be disjoint st $\dot{\bigcup}_{i=1}^{\infty} A_i \in S$.

Then $\dot{\bigcup}_{i=1}^{\infty} A_i = (a,b]$. Then, $\exists \{a_i\}_{i=1}^{\infty}$ st $b = a_1 > a_2 > a_3 \ldots$

and $(a_{i+1}, a_i] = A_j$.

~~Let $A_i = (p_i, q_i]$. Then we must have $p_i = b$ for some i. (For if not...). Argue similarly by induction.~~

~~Then $1 \geq \sum_{i=1}^{\infty} \mu(A_i) = \sum_{i=1}^{\infty} a_i - a_{i+1} = b-a$~~

This is not enough, I got it wrong. One needs Heine-Borel for this. See my 471 notes for this.

Dima Sinapova has an excellent article on the infinite ordinals and cardinals.
(Notices, vol 69, number 9 October 2022)
The ordinals are

$$0, 1, 2, 3, \ldots, \omega, \omega+1, \ldots$$

ω is the first ∞ ordinal #.

Well ordered set: S with a total ordering $<$ (or $\leq$) st every subset has a least element.
Ex: $\mathbb{N}$. $\mathbb{R}$ is not well ordered, ex $(1,2)$

Axiom of choice $\Leftrightarrow$ every set can be well-ordered (so you can just choose the least element)

Two well-ordered sets are order isomorphic if a bijection preserving their orderings.

Von-Neumann definition: Ordinals provide canonical representatives of each well-ordered set.

Ex: Consider $(\{1, 2, \ldots\}, <)$. The ordinal is ω.

In Russell's Principia, ordinals are as equivalence classes of well-ordered sets (but this equivalence class is too large to be defined under ZF. Cantor knows what that means)

Von Neumann just provides a canonical representative for each ordinal.

$0 = \phi$ (empty set)

$1 = \{\phi\}$ $2 = \{1\} = \{\phi, \{\phi\}\}$ ...

$\omega = \{1, 2, \ldots\}$

"Each ordinal is the well-ordered set of all smaller ordinals"

Formally: A set S is an ordinal iff S is strictly well-ordered with respect to set membership, and every element of S is also a subset of S.

$2 = \{\phi, \{\phi\}, \{\phi, \{\phi\}\}\}$ $\{\phi\} \subset \{\phi, \{\phi\}\}$

Is the class of all ordinals an ordinal? (Burali-Forti, similar to Russel's)

No it is not a set.

Each ordinal is also the set of ordinals before it. $\omega = \{1, 2, \cdots\} = \mathbb{N}$

In general $\beta = \{\alpha : \alpha < \beta\}$

So for each ordinal (viewed as a set) $(\beta, <)$ is a well order.

Transfinite recursion defines:

$$\beta + 1 = \beta \cup \{\beta\} \quad \text{(successor stage)}$$

Eg: ← successor stage

$3 = 2 \cup \{2\}, \ldots$

At the limit stage, one takes a union:

$\omega = \bigcup_{i \in \mathbb{N}} i$, $\omega + 1 =$

↑ axiom of union

$\underbrace{\omega, \omega+1, \ldots}_{\text{transfinite recursion}}$, $\omega + \omega$ ↖ limit stage

$\omega + \omega = \omega \cup \{\omega + n \mid n < \omega\}$

How many axioms in ZF? 8

Need an alphabet (countably ∞) to represent sets.

$\neg, \wedge, \vee$

$\forall, \exists$

$=, \in, ( )$

1) Extensionality 2) Regularity

3) Schema of specification (restricts what sets you can construct)

4) Axiom of pairing 5) Axiom of union (union over elements of a set exist).

6) axiom schema of replacement 7) axiom of ∞. 7 and with only many members.

8) Axiom of power set $\boxed{9}$ Choice / well-ordering

There was this book I read a long time ago, about how the empty set was at the core of everything, and about Russel, and Frege and Logicomix.

Cardinal #s: We count using one-one correspondences.

So anyway, many ordinals are countable: $|\omega| = |\omega+1| \cdots = |\omega \cdot \omega|$ and so on.

The first uncountable ordinal is ω_1 and by definition, this is not countable!

Then there is another jump at ω_2 and so on.

$\omega+1 = \{\omega\} \cup \{1, 2, \dots\}$ $\quad 1 = \{\phi\}$ $\quad 2 = \{1\} \cup \{\phi\} = \{\{\phi\}, \phi\}$

and so on.

Precisely, a cardinal is an ordinal κ which has no bijection to the ordinals below it.
The cardinality of a set $A = |\kappa|$ the cardinality of the ordinal (a set) equinumerous to it.
1-1 correspondence

Infinite cardinals: $\{\aleph_\alpha : \alpha \in \text{ordinal}\}$ $\quad \aleph_0$ is the cardinality of ω.
$\aleph_1$ is the cardinality of ω_1 and so on.

cartesian product

For $\kappa, \lambda \in \text{CARD}$, let $\kappa + \lambda = |\kappa \cup \lambda|$, $\quad \kappa \cdot \lambda = |\kappa \times \lambda|$

Easy to see $\quad \kappa + \lambda = \kappa \cdot \lambda = \max(\kappa, \lambda)$

Def: let κ^λ be the cardinality of the set of functions from λ to κ.

Ex: $f : \kappa \to \{0,1\} = 2$ encodes a subset of κ in the usual way. We have
$2^\kappa = |P(\kappa)|$

Gen. Continuum hypothesis: For all ∞ cardinals κ, (let κ^+ be the successor cardinal of κ)
$2^\kappa = |P(\kappa)| = \kappa^+$

In particular $\quad 2^{\aleph_0} = \aleph_1$

By Cohen's method of forcing, the continuum (2^ω) can have arbitrary high cardinality! MAD!

(Gödel has a minimal model of ZFC where CH holds, Cohen proved $\exists$ models where 2^ω has arbitrary size $\Rightarrow$ CH must be indep. of ZFC)

We need to show $\mu(a,b] = b-a$ is countably subadditive over $\mathcal{R}$ the collection of intervals. But instead lets show if

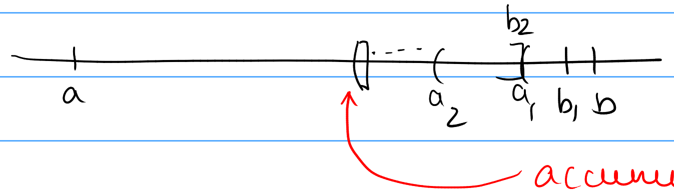
$$\star \quad (a,b] = \bigsqcup_{i=1}^{\infty} (a_i, b_i] \qquad \mu(a,b] = \sum_{i=1}^{\infty} b_i - a_i$$

Note $\exists\, b_i$ st $b_i = b$. If not $b_i < b \ \forall i$

$\Rightarrow b \notin (a_i b_i] \ \forall i$, contradicts $\star$.

So b is not a left accumulation point in b_i. Indeed one can see this is true of every a_i.

However, b_j can be a right accumulation point of either the set of $\{a_i\}$ or set of $\{b_k\}$. This is the issue.



Using transfinite recursion, we can define $F(\alpha) : \infty \to \mathbb{N}$ (class of ordinals)

$F(0) = j \ ; \ b_j = b$

Successor stage.

$F(\alpha+1) = k : b_k = a_{F(\alpha)}$

At the limit stage. β

let $p = \inf\limits_{\alpha<\beta} a_{F(\alpha)} \geq a$

$F(\beta) = k : b_k = p$

One can show this is well-defined. If $F(\beta) = a$ then we're done, and since F is a bijection to $\mathbb{N}$, β is countable.

Now prove by transfinite induction. $\forall \beta$ an ordinal

$$\sum_{\alpha < \beta} b_{F(\alpha)} - a_{F(\alpha-1)} = b - a_{F(\beta)}$$ at successor stage.

and then another argument for the limit stage.

3) Countable monotonicity: let $\{A_i\}_{i=1}^{\infty}$ be st $\bigcup_{i=1}^{\infty} A_i \supset B \in S$.
To show

$$\sum_{i=1}^{\infty} \mu(A_i) \geq \mu(B).$$

It's easy to write $\bigcup_{i=1}^{\infty} A_i = \bigsqcup_{j=1}^{\infty} F_j$ as a disjoint union.

where F_j are intervals. (Write $F_1 = A_1$, $F_2 = A_2 \setminus A_1, \dots$)
F_2 is obviously a disjoint union of intervals by the semiring property.

Thus

$$\mu\left(\bigsqcup_{j=1}^{\infty} B \cap F_j\right) = \mu(B) = \sum_{j=1}^{\infty} \mu(B \cap F_j) \leq \sum_{j=1}^{\infty} \mu(F_j) = \mu\left(\bigcup_{j=1}^{\infty} A_i\right)$$

From our countable additivity.

Then, let $S' = \{\bigsqcup_{k=1}^{n} A_k : A_k \in S\}$ is closed under relative complements

Then that's it! By Caratheodory-Hahn, $\exists$ a unique measure $\bar{\mu}$ on $\sigma(S)$ that extends μ.

pre-measure on Intervals $\longrightarrow$ Uniquely extend to pre-measure on Finite disjoint unions of intervals

$\longrightarrow$ Uniquely extend to meas. on σ-algebra.

Monotone class theorem :

Let $\Omega \in S$. There are many definitions of monotone class.

Khoshnevisan

S is a monotone class if $\forall \{A_i\}_{i=1}^{n} \in S$, st $A_i \subset A_{i+1}$ then $\bigcup_{i=1}^{\infty} A_i \in S$

The same holds for decreasing sequences of sets as well; i.e, $\bigcap_{i=1}^{\infty} A_i \in S$.

Le Gall : 1) If $A \subset B \in S$, then $B \setminus A \in S$

2) If $A_n \uparrow A$ (i.e., $\bigcup_n A_n = A$) then $A \in S$.

Note: Le Gall => Khoshnevisan since 1) => $A^c \in \Omega$

Le Gall's definition is the same as λ-system.

Lemma: let $\{M_\alpha\}_{\alpha \in A}$ be a collection of monotone classes for any set A.
Then

$$M = \bigcap_{\alpha \in A} M \quad \text{is a monotone class.}$$

let $F(A) = \{M : M \text{ is a monotone class containing } A\}$

Lemma: $MC(A) := \bigcap_{M \in F(A)} M$ is a monotone class.

Similarly let $\sigma(A)$ be the smallest σ-algebra containing A.

Any $\mathcal{F}$ that's a σ-algebra is also a monotone class.

$\Rightarrow \sigma(A) = \bigcap_{\mathcal{F} \supset \mathcal{A}} \mathcal{F} \supseteq MC(\mathcal{A})$ (since there could exist monotone classes that are not σ-algebras)

The opposite inclusion is the hard part.

- The monotone class lemma does this by showing $MC(A)$ is a σ-algebra and thus $MC(A) \supseteq \sigma(A)$.

So with additional structure on $\mathcal{A}$ we can show $MC(A) = \sigma(A)$.

Ex: Khoshnevisan says algebra

Monotone class theorem: If $\mathcal{A}$ is ________ $MC(A) = \sigma(A)$.

Le Gall says closed under finite intersection.

In fact Le Gall's book on Brownian Motion does this directly and the proof is nicely written. I found it hard to understand Khoshnevisan's proof but once I read Le Gall's, I was fine.

Le Gall's proof:

Note any σ-field is an MC. If $\mathcal{M}$ is an MC, it is a σ-field iff it is closed under finite intersectn.

Only the if part needs proof:

Suppose $A_1 \cap \dots \cap A_n \in \mathcal{M}$ $\forall \{A_i\}_{i=1}^n \in \mathcal{M}$.

Then by taking complements (recall $\Omega \setminus A_i \in \mathcal{M}$), it is closed under finite unions.
By MC property, its closed under countable unions.
$\Rightarrow$
Enough to show $MC(\mathcal{A})$ is closed under finite intersections.

Pf: Let $\mathcal{M}_A = \{ E \in MC(\mathcal{A}) : E \cap A \in MC(\mathcal{A}) \}$
Let $A \in \mathcal{A}$

1) Let $B \in \mathcal{A}$ then $B \cap A = \underbrace{B}_{\in MC(\mathcal{A})} \setminus \underbrace{(B \setminus A)}_{\in MC(\mathcal{A})} \in MC(\mathcal{A}) \Rightarrow \mathcal{A} \subset \mathcal{M}_A$

2) Let $E \subset F \in \mathcal{M}_A$ $\quad F \setminus E \cap A = \underbrace{F \cap A}_{\in \mathcal{M}_A} \setminus \underbrace{E \cap A}_{\in \mathcal{M}_A}$ by defn of E, F.

But $F \cap A$, $E \cap A \in MC(\mathcal{A})$ which is an MC!

3) Let $B_n \uparrow \in \mathcal{M}_A$, then $B_n \cap A \in MC(\mathcal{A})$ by defn.

$$\Rightarrow \bigcup_{n=1}^{\infty} B_n \cap A = \bigcup_{n=1}^{\infty} (B_n \cap A) \in MC(\mathcal{A}) \Rightarrow \bigcup_{n=1}^{\infty} B_n \in \mathcal{M}_A$$

So $\mathcal{M}_A$ is an MC containing $\mathcal{A}$, and $\mathcal{M}_A \supseteq MC(\mathcal{A})$

(in fact by defn $\mathcal{M}_A \subseteq MC(\mathcal{A})$ so they're equal)

CONSEQUENCE: If $A \in \mathcal{A}$, and $B \in MC(\mathcal{A})$. Since $B \in \mathcal{M}_A$
$\Rightarrow B \cap A \in MC(\mathcal{A})$

Now let $A \in MC(\mathcal{A})$ and consider $\mathcal{M}_A$. Let $B \in \mathcal{A}$.

$B \cap A \in MC(\mathcal{A})$ by part 1 $\Rightarrow B \in \mathcal{M}_A$. $\Rightarrow \mathcal{A} \subset \mathcal{M}_A$.

Rest of proof is identical.
- Let $E, F \in \mathcal{M}_A \Rightarrow F \cap A, E \cap A \in MC(\mathcal{A})$ which is an MC

$\Rightarrow F \cap A \setminus E \cap A \in MC(\mathcal{A}) \Rightarrow F \setminus E \cap \in MC(\mathcal{A}) \Rightarrow F \setminus E \in \mathcal{M}_A$

- Similar proof of $B_n \uparrow$.

So $\mathcal{M}_A = MC(\mathcal{A})$ and $MC(\mathcal{A})$ is closed under finite intersections.

let μ be another meas. st $\mu(a,b) = b-a$. let λ be Lebesgue meas. Let

$$G = \{ E : \mu(E) = \nu(E) \}.$$

Then $G \supset \{(a,b) : a<b\}$ and it's a monotone class since μ & ν are measures: restrict first to $[-n,n]$. Then $\mu, \nu < \infty$.

1) $[-n,n] \in G$

2) $E \subset F$, $\mu(F \setminus E) = \mu(F) - \mu(E) = \nu(F) - \nu(E)$
(need finiteness for this. Otherwise if $\mu(E), \mu(F) = \infty$ we're doomed)

3) $B_n \uparrow$ is quite similar.

So $G \supset MC(\{(a,b) : a<b, (a,b) \subset [-n,n]\}) = \mathcal{B}([-n,n])$

Then for $\mathbb{R}$, and $E \in \mathcal{B}(\mathbb{R})$, $\mu(E) = \lim_{n\to\infty} \mu(E \cap [-n,n])$ and we're done.

→ Typically, one would prove Caratheodory using the MC lemma in a probability book. This is done in Khoshnevisan, e.g.

→ It is useful in the "uniqueness" part of the Caratheodory theorem. But Khoshnevisan also uses it elsewhere.

→ Royden does not need to use the Monotone Class theorem anywhere, and in fact showing

$$\mu : S \longrightarrow [0, \infty)$$

$\mu = \mu^*$ requires countable subadditivity of μ

$S \in \{\text{Measurable sets}\}$ follows from closure under relative complements.

The only bit where he needs something extra is to show uniqueness, for which he uses the fact that it's enough to show uniqueness on $S_{\sigma\delta}$.

Khoshnevisan's proof of the MC lemma

Let $\mathcal{A}$ be an algebra.

$C_1 = \{E \in \sigma(\mathcal{A}) : E^c \in MC(\mathcal{A})\}$

Claim: C_1 is a monotone class containing $\mathcal{A}$

Pf: Let $B \in \mathcal{A}$, then $B^c \in \mathcal{A}$ since $\mathcal{A}$ is an λ-system

$\Rightarrow B \in C_1$

Let $\{B_n\} \in C_1$ be increasing. $\left(\bigcup_{n=1}^{\infty} B_n\right)^c = \bigcap_{n=1}^{\infty} B_n^c \in MC(\mathcal{A})$ since it's an MCl.

CONSEQUENCE: if $E \in MC(\mathcal{A}) \subseteq C_1$ then $E^c \in MC(\mathcal{A})$.

$\Rightarrow$ $MC(\mathcal{A})$ is closed under complements.

Next, to show $MC(\mathcal{A})$ is closed under finite unions. This is by bootstrapping.

Let $C_2 = \{E \in MC(\mathcal{A}) : \forall F \in \mathcal{A} \; E \cup F \in MC(\mathcal{A})\}$

Claim: C_2 is an MC containing $\mathcal{A}$:

- If $B \in \mathcal{A}$, then $F \cup B \in MC(\mathcal{A}) \; \forall F \in \mathcal{A}$ (it's an algebra)

- Let $B_n \uparrow$ in C_2. Then for any $F \in \mathcal{A}$, $B_n \cup F \in MC(\mathcal{A})$ (by defn of C_3)

$$\Rightarrow \bigcup_{n=1}^{\infty} B_n \cup F = \underbrace{\bigcup_{n=1}^{\infty} (B_n \cup F)}_{\text{in every MC containing } \mathcal{A}.} \in MC(\mathcal{A})$$

using monotone class property and we're done.

CONSEQUENCE

Fix $B \in MC(\mathcal{A})$. Then since $\mathcal{B} \subset MC(\mathcal{A}) \subseteq C_2$, $B \cup F \in MC(\mathcal{A})$ $\forall F \in \mathcal{A}$

$\Rightarrow$ $\forall B \in MC(\mathcal{A})$, $F \in \mathcal{A}$, $B \cup F \in MC(\mathcal{A})$

Finally, let

$C_3 = \{ E \in MC(\mathcal{A}) : \forall F \in MC(\mathcal{A}), E \cup F \in M(\mathcal{A}) \}$

Claim: C_3 is an MC containing $\mathcal{A}$

Pf: Let $B \in \mathcal{A}$. Then by previous, $B \cup F \in MC(\mathcal{A})$ $\forall F \in MC(\mathcal{A})$. (Bootstrapping).

$\Rightarrow$ $B \in C_3$

- Let $\{B_i\}_{i=1}^{\infty} \in C_3$ be increasing. Then $B_i \cup F \in MC(A)$ $\forall F \in MC(A)$

$\left(\bigcup_{i=1}^{\infty} B_i\right) \cup F = \bigcup_{i=1}^{\infty} B_i \cup F \in MC(A)$ since $B_i \cup F$ must be increasing.

Similarly if B_i is decreasing, $\bigcup_{i=1}^{\infty} B_i^c \in MC(A)$ and by the C_1-step,

$\bigcap_{i=1}^{\infty} B_i \in MC(A)$.

So if $B \in MC(A) \subseteq C_3 \Rightarrow B \cup F \in MC(A)$ $\forall$ $F \in MC(A)$.

Π-λ Theorem (Durrett online version 5a, page 414, Theorem A.1.4)

The π-λ theorem is due to Dynkin. This is simply another name for the Monotone class lemma.

π-system: $\mathcal{P}$ is a π-system if it is closed under finite intersection (Even less structure than a semi-ring!)

λ-system: Same as monotone class of sets: $\mathcal{L}$ is a λ-system if a) $\Omega \in \mathcal{L}$ b) If $A \subset B$, then $B \setminus A \in \mathcal{L}$ c) closed under monotone increasing limits. (superset relative complement closed)

Theorem: If $\mathcal{P}$ is a π-system and $\mathcal{L}$ is a λ-system containing $\mathcal{P}$, then $\sigma(\mathcal{P}) \subset \mathcal{L}$

☆ Show that this is equivalent to the monotone class lemma. Hint: consider $M(\mathcal{P})$ the smallest λ-system containing $\mathcal{P}$

☆ Suppose $X_1, \ldots, X_n$ are rvs such that

$$\mathbb{P}(X_1 \le x_1, \ldots, X_n \le x_n) = \prod_{i=1}^{n} \mathbb{P}(X_i \le x_i)$$

Show that $X_1, \ldots, X_n$ are independent.

Note: Khoshnevisan defines a monotone class as follows: closed under increasing countable unions & decreasing countable intersections.

Le Gall defines a monotone class as
1) $\Omega \in M$ 2) closed under relative complements
3) closed under increasing unions.

Are the two equivalent? Obviously Le Gall $\Rightarrow$ Khoshnevisan.
Clearly not! ☆ Come up with an example.

Then why are they more or less the same? Recall their statements:

(Le Gall): let $\mathcal{C}$ be a π-system (closed under intersections). Then $M(\mathcal{C}) = \sigma(\mathcal{C})$ (the relative complement closure is hidden in the definition of the monotone class!

(Khoshnevisan): let A be an algebra, Then $M(\mathcal{C}) = \sigma(\mathcal{C})$.

The π-λ theorem or Monotone class theorem a'la Durrett / Le Gall are easier to use in general.

Completions :

Claim: Let E be set such that $\mu^*(E) = 0$. Then E is measurable!

$\mu^*(A \cap E) \le \mu^*(E) = 0$ To show $\mu^*(A) = \mu^*(A \cap E^c)$

We have $\mu^*(A) \geq \mu^*(A \cap E^c)$, and $\mu^*(A) \le \mu^*(A \cap E) + \mu^*(A \cap E^c)$

So if $\mathcal{F}$ is the set of measurable sets produced by Caratheodory extension, then it contains all subsets of measure 0 sets. Thus it is complete!

Theorem: Given a measure space $(\Omega, \mathcal{F}, \mu)$, then $\exists$ a complete σ-algebra $\mathcal{F}' \supseteq \mathcal{F}$ and a measure μ' on $(\Omega, \mathcal{F}')$ such that μ and μ' agree on $\mathcal{F}$.

Pf: (Instead of Davar's proof, consider proof by Caratheodory extension!)

Comments:

1) Khoshnevisan proves Caratheodory using the MC-theorem.
2) Durrett proves the $\pi - \lambda$ theorem and then proves Caratheodory.
3) Royden does not use the MC-Theorem. In fact, he uses a very useful approximation property of outer measure.

Prop: Let μ be a set function on S and let $\bar{\mu}$ be the Caratheodory measure induced by μ. Let E be st $\mu^*(E) < \infty$. Then $\exists$ $A \subset X$, $A \in S_{\sigma\delta}$ $E \subseteq A$ and $\mu^*(E) = \mu^*(A)$.

If E is assumed to be measurable, and so are all the sets in S, then $\bar{\mu}(A \setminus E) = 0$.

Recall S_σ = countable unions of sets in S

$S_{\sigma\delta}$ = countable intersections of sets in S_σ.

"$S_{\sigma\delta}$ is rich enough"

Good exercises: (3.3), (3.4), (3.9), (3.10)

Royden, for example, cares about completeness for the following reason:
We say a property holds a.e. if $\exists X_0$ st $\mu(X \setminus X_0) = 0$ and the property holds on X_0

2. Suppose $(X, \mathcal{F}, \mu)$ is not complete. Let $E \subset X$ st $E \notin \mathcal{F}$ but $E \subset A \in \mathcal{F}$, where $\mu(A) = 0$. Let $f = 0$ on X, and $g = 1_E$ on X.
Show
i) $f = g$ a.e
2) g is not measurable.

$f = g$ a.e.: $f = g$ on $X \setminus A$ since $E \subset A$. But $\mu(A) = 0$.

3. Assume $(X, \mathcal{F}, \mu)$ is not complete.
Show that there is a sequence of functions f_n st f_n converges to an f a.e. X and f is not measurable.

Let $f_n = 0$ be the constant sequence. Then $f_n \to 1_E$ as above a.s.
Is that what they mean?

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence. Then $S = \sup_n f_n$ is measurable. Why is that?
$S \le x \iff f_n \le x \ \forall n$
So $S^{-1}(-\infty, x] = \bigcap_n f_n^{-1}(-\infty, x]$ which is measurable.
Thus $\overline{\lim} f_n = \inf_m \sup_{n \ge m} f_n$ is measurable. An identical argument gives $\underline{\lim} f_n$ is also measurable. So if $\lim f_n$ exists then $\overline{\lim} f_n$ and $\underline{\lim} f_n$ also exist.

However, is it possible to construct a f_n st $\overline{\lim} f_n = \underline{\lim} f_n$ outside of a set of measure 0. I don't see how its possible in general: $\overline{\lim} f_n = \underline{\lim} f_n$ on some measurable set E. Define f to be normal outside of $X \setminus E$. Maybe just set it be 0. Then f must be measurable.

I think this must be the ambiguity: f can be weird outside on the bad set A. Maybe one should then say there is a measurable <u>version</u> of $\lim_{n\to\infty} f_n$.

To me this is mostly semantics. In fact, if f_n indeed converges everywhere, then $\lim_{n\to\infty} f_n$ is measurable.